Hyperbolic Substitutions for Integrals

Khristo N. Boyadzhiev
Ohio Northern University
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In order to evaluate integrals containing radicals of the form

$$\sqrt{a^2 \pm x^2} \text{ and } \sqrt{x^2 - a^2}, \quad (a > 0),$$

most calculus textbooks use the trigonometric substitutions

1 For $\sqrt{a^2 - x^2}$ set $x = a\sin\theta$, or $x = a\cos\theta$;

2 For $\sqrt{a^2 + x^2}$ set $x = a\tan\theta$;

3 For $\sqrt{x^2 - a^2}$ set $x = a\sec\theta$.

However, while the substitution in 1 works fast, sometimes the substitutions in 2 and 3 require longer computations. We shall demonstrate here that in these two cases it is more natural to use the hyperbolic substitutions

2* For $\sqrt{x^2 + a^2}$ set $x = a\sinh t$, (1)

3* For $\sqrt{x^2 - a^2}$ set $x = a\cosh t$, (2)

where $-\infty < t < +\infty$.

We also use the basic identity for hyperbolic functions

$$\cosh^2 t - \sinh^2 t = 1 , \tag{3}$$

thus

$$\sqrt{x^2 + a^2} = \sqrt{a^2(\sinh^2 t + 1)} = a\cosh t ,$$

and

$$\sqrt{x^2 - a^2} = \sqrt{a^2(\cosh^2 t - 1)} = a\sinh t .$$

When returning to the original variable x, in order to simplify the final result it is convenient to use the equations

$$\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1}), \; -\infty < z < +\infty , \tag{4}$$

$$\cosh^{-1} z = \ln(z+\sqrt{z^2-1}),\ 1 \le z < +\infty , \tag{5}$$

$$\tanh^{-1} z = \frac{1}{2}[\ln(1+z) - \ln(1-z)]\ ,\ -1 < z < 1 . \tag{6}$$

and also the identities

$$1 - \tanh^2 t = \frac{1}{\cosh^2 t}\ , \tag{7}$$

$$\coth^2 t - 1 = \frac{1}{\sinh^2 t} \tag{8}$$

$$\cosh^2 t\ =\ \frac{1}{2}(\cosh 2t + 1), \tag{9}$$

$$\sinh^2 t\ =\ \frac{1}{2}(\cosh 2t - 1)\ , \tag{10}$$

$$\sinh 2t = 2 \sinh t \cosh t . \tag{11}$$

Examples.

1. Evaluate

$$F(x) = \int \frac{\sqrt{x^2-3}}{x^2}\, dx .$$

Assuming without loss of generality that $x > 0$, we set $x = \sqrt{3}\cosh t$ to obtain

$$\sqrt{x^2-3} = \sqrt{3}\sinh t\ ,\ dx = \sqrt{3}\sinh t\, dt\ ,$$

$$F = \int \frac{\sinh^2 t}{\cosh^2 t}\, dt = \int \frac{\cosh^2 t - 1}{\cosh^2 t}\, dt$$

$$= t - \tanh t + C .$$

Therefore,

$$F(x) = \cosh^{-1}\frac{x}{\sqrt{3}} - \tanh(\cosh^{-1}\frac{x}{\sqrt{3}}) + C ,$$

and according to (5) and (7)

$$F(x) = \ln(x + \sqrt{x^2 - 3}) - \frac{\sqrt{x^2 - 3}}{x} + C.$$

2. Evaluate

$$F(x) = \int \sqrt{x^2 + 4}\, dx.$$

We set $x = 2\sinh t$ to get

$$F = 4\int \cosh^2 t\, dt = 2\int (\cosh 2t + 1)\, dt$$

$$= \sinh 2t + 2t + C,$$

and in view of (11)

$$F(x) = \frac{x}{2}\sqrt{x^2 + 4} + 2\ln(x + \sqrt{x^2 + 4}) + C.$$